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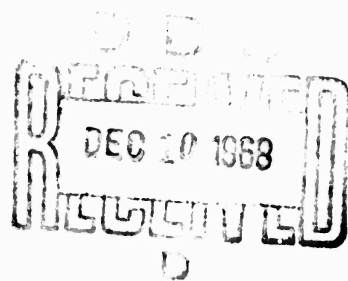
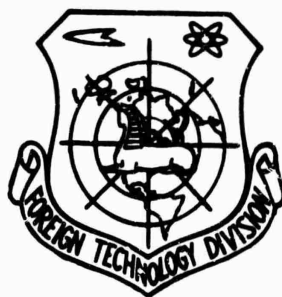
FOREIGN TECHNOLOGY DIVISION



THE USE OF ELECTRONIC DIGITAL COMPUTING MACHINES TO ANALYZE
THE STABILITY OF COMPLICATED ENERGY SYSTEMS

by

V. F. Kurov



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ABSTRACT

→ The possible methods of constructing regions of stability of a linear automatic control system in the space of the control parameters by using electronic digital computing machines and also methods for obtaining the coefficients of the characteristic equation

$$S(p) = \sum_{n=0}^{\infty} a_n p^n = 0,$$

are considered. The methods of constructing regions of stability that are considered in the article are based on application of algebraic criteria for stability. The use of Newton's interpolation method is extremely convenient when one is programming an electronic digital computing machine since it makes it possible to use the standard programs of linear algebra. This method also compares favorably with the other methods by making fewer demands on the memory of the machine at the cost of a row-by-row formation of the matrix of the transformation. Orig. art. has: 1 figure, 2 tables, and 14 formulas.

THE STABILITY OF COMPLICATED ENERGY SYSTEMS

V. F. Kurov

The original equations for the disturbed motion of complicated automatically controlled systems can be represented in the following matrix form:

$$pX = AX; X = \begin{vmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{vmatrix}; A = \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \cdot & \cdot & \cdot \\ a_{n1} & \dots & a_{nn} \end{vmatrix}, \quad (1)$$

where the differential equation (1) is the first approximation of the mathematical model of the system

[illegible]

Thus, when one takes into account the basic determining factors, the solution of the question regarding stability "in the small" (static stability) reduces to analysis of the

system (1) of linear differential equations describing the electromechanical and electromagnetic transfer processes. As usual, the characteristic equation of the system in question has a rather high order and can be represented in the form

$$S(p) = \sum_{s=0}^n a_s p^{n-s} = 0, \quad (3)$$

where $a_s = f_s(k_1, \dots, k_n)$, where, in turn, k_1, \dots, k_n are the control parameters.

The difficulty in analyzing such a system is due both to the calculation of the coefficients of the characteristic equation and to the application of the stability criteria themselves, which are quite laborious for equations of high degree and necessitate spending a considerable amount of time for their verification. The application of electronic digital computing machines requires the construction of optimal algorithms that will enable one to answer the above-mentioned questions in the least amount of time and, in some cases, with the least demands on the memory of the machine. Here, it is desirable to use ready-made standard programs of linear algebra.

In the present article, we shall consider the possible methods of constructing regions of stability of a linear automatic-control system in the space of the control parameters by using electronic digital computing machines and also methods for obtaining the coefficients of the characteristic equation.

The coefficients of the characteristic equation (3) are independent of the time but they depend on the set of control parameters k_1, \dots, k_n , which vary in some closed bounded region \bar{K} . The investigation problem consists in finding a closed region $\bar{G} \subset \bar{K}$ in which the roots of the characteristic equation will have only negative real parts.

In searching for such a region (the region of stability), we usually use the methods of a D-partition [1, 2]. If the number of control parameters does not exceed 2, then the boundary of the region of stability can be obtained in parametric form in a comparatively simple manner [3].

Although it is possible in theory to apply the method of a D-partition for a larger number of parameters [4], the calculations in such a case involve considerable difficulties, especially if the degree of equation (3) is high. The analysis of the results obtained is considerably complicated. Although use of an electronic digital computing machine alleviates the computational difficulties, the necessity of changing the frequency over a wide range, the necessity of choosing the optimal step, etc. render uneconomical the use of such a machine.

Therefore, the tendency to use directly the criteria of Routh, Hurwitz and Yu. I. Naymark [1, 2] directly to analyze the stability and to find boundaries of stability in the region of the control parameters [5, 6] is completely justified.

The methods of constructing regions of stability that are considered in the article are based on application of algebraic criteria for stability. Therefore, as a preliminary, let us look at the possible methods of calculating the coefficients of the characteristic equation.

Methods of Obtaining the Coefficients of the Characteristic Equation

The characteristic equation of disturbed motion of a complicated energy system can usually be represented in the form of the determinant of an n -th-degree matrix polynomial [7, 8].

$$S_1(p) = |A_0 p^r + A_1 p^{r-1} + \dots + A_r| = 0, \quad (4)$$

where A_0, A_1, \dots, A_r are square matrices of order m .

To set up a program for calculating the stability and the electro-mechanical transfer processes of a complicated energy system on a electronic digital computing machine, it is necessary to represent this determinant in the form of a scalar polynomial. To this end, we can use algorithms of two basic forms. The first of these includes algorithms based on linearized equations of a disturbed motion, which are, as a preliminary, reduced to a form solved for the derivatives. Then, the coefficients of the characteristic equation can be found by expanding the determinant of the "secular" equation.

The second form of algorithms, which do not involve the necessity of reducing the original equations to normal form, was developed by L. V. Tsukernik [9-11].

The methods of representing the determinant (4) in the form of a polynomial include the direct expansion of it by means of an expansion in terms of the elements of any row or column and also the reduction of the determinant (4) to a triangular form. However, these methods require either a large number of operations or a complicated program.¹

Let us look at some of the possible cases of the obtaining of the coefficients of the characteristic equation.

1. Transformation of Equation (4) to the Form $|B - \lambda E| = 0$.

To obtain the characteristic polynomial, we can use an artificial device based on the theorems of matrix algebra.

An m th-order determinant whose elements are r th-degree polynomials can always be transformed into a determinant of order mr whose elements are linear in p provided the matrix of the coefficients of p^r is nonsingular.

Suppose that the given equation is of the form (4). Then, its roots, as one can verify directly, are equal to the characteristic number of the matrix

$$B = \left\| \begin{array}{cccccc} -A_0^{-1}A_1 & -A_0^{-1}A_2 & \dots & -A_0^{-1}A_{r-1} & -A_0^{-1}A_r & \\ E & 0 & \dots & 0 & 0 & \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & E & 0 & \end{array} \right\| \quad (5)$$

where A^{-1} is the inverse of the matrix A_0 and E is the unit matrix of m th order.

The characteristic equation of the matrix (5) can be found by means of Danilevskiy's method. Because of the special form of the matrix (5), the calculating process is carried out considerably more rapidly than with other methods [12].

The characteristic equation of the matrix (5) can be represented in the form

$$S(\lambda) = (-1)^n \sum_{s=0}^n a_s \lambda^{n-s} = 0. \quad (5a)$$

If we replace λ with p in equation (5a), we obtain the characteristic equation of the system in question.

However, in a number of cases, the matrix A_0 in equation (4) may not have an inverse. If we replace p with $1/p_1$ in equation (4), we obtain the equation

$$|A_r p_1^r + A_{r-1} p_1^{r-1} + \dots + A_0| = 0, \quad (6)$$

the roots of which are obviously equal to the reciprocals of the characteristic numbers of the matrix

$$B_1 = \begin{vmatrix} -A_r^{-1} A_{r-1} & -A_r^{-1} A_{r-2} & \dots & -A_r^{-1} A_1 & -A_r^{-1} A_0 \\ E & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & E & 0 \end{vmatrix} \quad (7)$$

where A_r^{-1} is the inverse of the matrix A_r , which we can always make nonsingular by choosing suitable values for the control parameters.

The characteristic equation of the matrix (7) can also be represented in the form of equation (5a). Then, the characteristic equation of the automatic-control system is obtained from equation (5a) by replacing λ with $1/p$.

2. Application of the Method of Interpolation to Equation (4).

Since the function $S_1(p)$ in equation (4) is a polynomial of degree not exceeding n , the interpolational polynomial $S(p)$ must be identically equal to $S_1(p)$. This follows from the theorem on the uniqueness of the interpolational polynomial.

Expansion of the m th-order determinant (4) yields a polynomial of degree $n \leq mr$ in p . This polynomial contains $n + 1$ coefficients of the different powers. These coefficients can be determined if we know the values of $S_1(p)$ for $n + 1$ values of p . To do this, we can use the interpolation formula or we can solve a system of linear equations for the coefficients. If the difference between two successive values of p is constant,

then, to obtain a polynomial expression, we can use the first interpolation formula of Newton [12].

$$S(p) = S(\delta) + \frac{x}{1!} \Delta S(\delta) + \frac{x(x-1)}{2!} \Delta^2 S(\delta) + \dots + \frac{x(x-1)\dots(x-n+1)}{n!} \Delta^n S(\delta), \quad (8)$$

where $x = (p - \delta)/h$ and the differences $\Delta^k S(\delta)$ can be taken from the following Table:

$S(\delta)$			
$S(\delta + h)$	$\Delta S(\delta)$		
		$\Delta^2 S(\delta)$	
$S(\delta + 2h)$	$\Delta S(\delta + h)$		$\Delta^3 S(\delta)$
		$\Delta^2 S(\delta + h)$.
$S(\delta + 3h)$	$\Delta S(\delta + 2h)$.	.
.	.	.	.
.	.	.	.
$S(\delta + nh)$.	.	.

where $\Delta S(\delta) = S(\delta + h) - S(\delta)$, $\Delta^2 S(\delta) = \Delta S(\delta + h) - \Delta S(\delta)$, etc.

The k th-order difference of a k th-degree polynomial is equal to zero [sic], which enables us to determine immediately the order of the characteristic equation.

Formula (8) does not immediately yield a polynomial form since each term in it is a polynomial in p . Let us consider the interpolational polynomial (8) arranged in increasing powers of $p - \delta$, so that the coefficients can be calculated immediately with the aid of Table 1.² To do this, we transform it to the form

$$S(p) = \sum_{k=0}^n b_k (p - \delta)^k = b_0 + b_1 (p - \delta) + b_2 (p - \delta)^2 + \dots + b_n (p - \delta)^n, \quad (9)$$

where

$$b_k = \frac{1}{h^k} \sum_{s=k}^n \rho_{k(s)} \Delta^s S(\delta), \quad (k = 1, 2, \dots, n), \quad b_0 = S(\delta). \quad (10)$$

For convenience in making the calculations, we need to set $\delta = 0$ and $h = 1$, which leads to a certain simplification of the expressions (9) and (10).

Table 1

The Value of the Coefficients of the Matrix P

	1	2	3	4	5	6	7	8
1	+1000000+01	+5000000+00	+16.6667+00	+4166667-01	+8333333-02	+1388889-02	+1984127-03	+2480159-04
2	-5000000-00	-5000000+00	-2500.000+00	-8333333-01	-2083333-01	-4166667-02	-6944444-03	-9920635-04
3	-3333333-00	+4583333+00	+2916667+00	+11803556+00	+3472222-01	+7986111-02	+1504630-02	+2397486-03
4	-2500000-00	-4166667+00	-3125000+00	-1458333+00	-4861111-01	-1743634-01	-2604167-02	-4546957-03
5	-2000000-00	+38035556+00	+3222222+00	+1678819+00	+6186342-01	-2259838-01	+3952546-02	+7461833-03
6	-1666667+00	-3500000+00	-3256914+00	+1994268+00	-7421875-01	-2784862-01	-5506366-02	-1112351-02
7	-1428571+00	-3241071+00	-3231677+00	+2197447+00	+8560130-01	-3309289-01	+7225001-02	+9072708-02
8	-1250000-00	-3019842+00	+3255180+00	-1854167+00	-9602417-01	+2784862-01	-9072708-02	+1548963-02
9	-1111111+00	-2828968+00	+3195040+00	+1994268+00	+1035411+00	-3309289-01	+1101904-01	-2050552-02
10	-1000000-00	-2628968+00	+3150678+00	+2197447+00	+1142223+00	+3826777-01	+1303857-01	+2611082-02
11	-9090909-01	-2516561+00	-3150678+00	-2270772+00	-1221417+00	-4333140-01		
12	-8333333-01	-2387085+00	+3101899+00	+2330139+00	+1221417+00	+4125718-01		
13	-7692307-01	-2271524+00	+3050842+00	-2378186+00	-1293715+00			
14	-7142857-01	-2167708+00	+2998887+00	+2416980+00				
15	-6666667-01	+2073893+00	-2946939+00					
16	-6250000-01							
9	+2255732-05	+2755731-06	+2505210-07	+2087675-08	+16.59903-09	+1147073-10	+7647151-12	+4779472-13
10	-1240079-04	-1377866-05	-1377866-06	-1252605-07	-1043837-08	-8029518-10	-5735363-11	
11	+330687-04	+4018774-05	+4363242-06	+4279734-07	+3527405-07	+3144895-09		
12	-6820436-04	-8956126-05	-1044881-05	-1096629-06	-1043837-08			
13	+1203566-03	+1691330-05	-2102776-05	-2341763-06				
14	-1912133-03	-2853330-05						
15	+2817304-03	+4435812-05						
16	-3922817-03							

Remark: Every number consists of the following elements: the sign of the mantissa, the mantissa, the sign of the order, the order. For example -9072708 - 02 = -0.009072708.

If δ is chosen different from zero, the shift to the degree p instead of $(p - \delta)$ is easily carried out with the aid of Horner's scheme or in accordance with the scheme shown below:

$$\begin{aligned} a_0 &= a_n b_n \\ a_1 &= a_{n-1} b_{n-1} + a_{n-1} b_n \delta \\ a_2 &= a_{n-2} b_{n-2} + a_{n-2} b_{n-1} \delta + a_{n-2} b_n \delta^2 \\ &\dots \dots \dots \\ a_{n-1} &= a_{11} b_1 + a_{12} b_2 \delta + \dots + a_{1n} b_n \delta^{n-1} \\ a_n &= a_{00} b_0 + a_{01} b_1 \delta + \dots + a_{0n} b_n \delta^n. \end{aligned}$$

Or, in the general case,

$$a_{n-k} = \sum_{v=0}^{n-k} a_{k(v)} b_{k+v} \delta^v; \quad k = 0, 1, 2, \dots, n. \quad (11)$$

Here,

$$\begin{aligned} a_{ij} &= (-1)^j, \quad a_{ij} = 0, \quad i > j, \\ a_{ij} &= (-1)^{i+j} \sum_{k=i-1}^{j-1} |a_{(i-1)k}| = (-1)^{i+j} \sum_{s=i-1}^j |a_{s(j-1)}|, \\ &\quad i \leq j, \quad (i, j = 0, 1, \dots, n). \end{aligned}$$

The values of the coefficients a_{ij} for equations of the first sixteen orders are shown in Table 2.

In the case of constant differences between successive values of p , Newton's first interpolation formula becomes unsuitable. In this case, we can use Lagrange's formula or the method of undetermined coefficients.

3. The Application of the Method of Undetermined Coefficients.

Let us look at an expression of the form

$$S(p) = a_n + a_{n-1}p + \dots + a_1 p^{n-1} + a_0 p^n.$$

If we choose $n + 1$ values of p , then to determine the coefficients a_i , we need to solve $r + 1$, linear equations (or n linear equations in the case in which one of the values of p is zero, in which case $S(0) = a_n$)

$$\begin{aligned} a_n + a_{n-1}p_i + a_{n-2}p_i^2 + \dots + a_1 p_i^{n-1} + a_0 p_i^n &= S(p_i), \\ (i &= 1, 2, \dots, n + 1), \end{aligned} \quad (13)$$

Table 2
The Value of the Coefficients n_{ij}

$i \backslash j$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1
1		1	-2	3	-4	5	-6	7	-8	9	-10	11	-12	13	-14	15	-16
2			1	-3	6	-10	15	-21	28	-36	45	-55	66	-78	91	-105	120
3				1	-4	10	-20	35	-56	84	-120	165	-220	286	-364	455	-560
4					1	-5	15	-35	70	-126	210	-330	495	-715	1001	-1365	1820
5						1	-6	21	-56	126	-252	462	-792	1287	-2002	3003	-4368
6							1	-7	28	-84	210	-462	924	-1716	3003	-5005	8008
7								1	-8	36	-120	330	-792	1716	-3432	6435	-11440
8									1	-9	45	-165	495	-1287	3003	-6435	12870
9										1	-10	55	-220	715	-2002	5005	-11440
10											1	-11	66	-286	1001	-3003	8008
11												1	-12	78	-364	1365	-4368
12													1	-13	91	-455	1820
13														1	-14	105	-560
14															1	-15	120
15																1	-16
16																	1

or, in matrix form,

$$P \cdot a = S, \quad (13a)$$

where

$$P = \begin{vmatrix} 1 & p_1 & p_1^2 & \dots & p_1^n \\ 1 & p_2 & p_2^2 & \dots & p_2^n \\ 1 & p_3 & p_3^2 & \dots & p_3^n \\ \dots & \dots & \dots & \dots & \dots \\ 1 & p_n & p_n^2 & \dots & p_n^n \end{vmatrix} \quad a = \begin{vmatrix} a_n \\ a_{n-1} \\ a_{n-2} \\ \vdots \\ a_0 \end{vmatrix} \quad S = \begin{vmatrix} S(p_1) \\ S(p_2) \\ S(p_3) \\ \vdots \\ S(p_n) \end{vmatrix} \quad (14)$$

The system of equations (13) with $p_i = \delta + ih$ can be reduced by means of elementary transformations [12] to the equivalent step form

$$P'^{-1} H^{-1} b = \Delta S(\delta), \quad b_0 = \left| \sum_{i=0}^r A_i \delta^{r-i} \right|,$$

where the coefficients b determine the characteristic equation $\sum_{i=0}^n b_i (p - \delta)^i = 0$. The imaginary axis in the complex plane of the roots of this equation is displaced to the right (resp. to the left) for positive (resp. negative) δ with respect to the imaginary axis of the complex plane of the roots of the equation

$$\sum_{i=0}^n a_i p^{n-i} = 0.$$

Here, P'^{-1} is the matrix inverse to P' , $H = [h \ h^2 \ \dots \ h^n]$ is a diagonal matrix, $\Delta S(\delta) = (\Delta S(\delta) \ \Delta^2 S(\delta) \ \dots \ \Delta^n S(\delta))$ and $b = (b_1 \ b_2 \ \dots \ b_n)$ are the column matrices of the corresponding quantities.

One can easily see that finding the coefficients of the characteristic equation by using the interpolation method (10) is, by its nature, a more rational procedure for solving the problem than the method of undetermined coefficients in the case of equally spaced values of p .

To solve the system of equations (13a), we can use the familiar methods

$$a = P^{-1} S. \quad (13b)$$

In the case in which the interval between adjacent values of p is equal to 1 and $p_1 = 0$, the system of equations can be written in the form

$$Pa = \begin{vmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2^2 & \dots & 2^n \\ \dots & \dots & \dots & \dots & \dots \\ 1 & n & n^2 & \dots & n^n \end{vmatrix} \cdot \begin{vmatrix} a_n \\ a_{n-1} \\ a_{n-2} \\ \vdots \\ a_0 \end{vmatrix} = \begin{vmatrix} S(0) \\ S(1) \\ S(2) \\ \vdots \\ S(n) \end{vmatrix} \quad (13c)$$

The matrix P in the expression (13c) is independent of the numerical values of $S(p_i)$. We can calculate in advance the inverse matrix P^{-1} , and then the solution of the system (13c) reduces to multiplying matrices in accordance with the expression (13b).

Figure 1 shows the number of operations necessary to calculate the characteristic polynomial by different methods for $r = 2$, depending on the order of the matrix m .

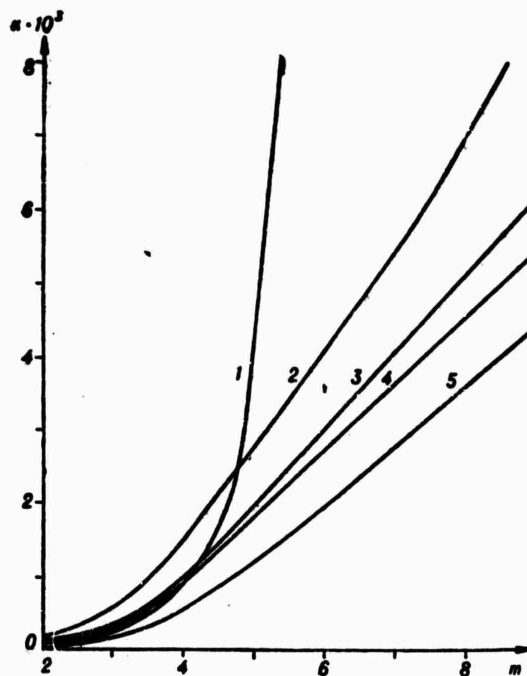


Fig. 1. The dependence of the number of computational operations on the order of the determinant.

- 1) Direct expansion; 2) The method of undetermined coefficients; 3) The method of the interpolation formula; 4) Transformation to diagonal form; 5) The method of undetermined coefficients with a constant step when there is an inverse matrix.

As one can see from Fig. 1, direct expansion yields the smallest number of operations if the determinant is of fourth order; the interpolational method and transformation to diagonal form are preferable for higher orders. The interpolational method, however, is the most convenient one even if it requires somewhat more operations. This method enables us, by the nature of the change in the function of the differences, to predict the order of the characteristic equation. The values of the function can be calculated fairly easily with the aid of an electronic digital computing machine with the minimal number of input data. Although the method of undetermined coefficients requires fewer operations in the case of equally spaced values of p than does the method of transformation to diagonal form, the necessity of finding the inverse of a matrix of high order makes this method less desirable when one is using an electronic digital computing machine.

It is convenient to employ the method of transformation to diagonal form when one has found even "inexact" though easily visualized regions of stability [13]. (By "inexact" regions, we mean regions somewhat smaller than the usual regions of stability defined by the Routh-Hurtz inequalities.)

Thus, the use of Newton's interpolation method is extremely convenient when one is programming an electronic digital computing machine since it makes it possible to use the standard programs of linear algebra. In addition, this method compares favorably with the others in that it makes fewer demands on the memory of the machine at the cost of a row-by-row formation of the matrix of the transformation.

Let us indicate the order in which we obtain the characteristic polynomial if the control parameters k_1, \dots, k_n appear linearly in equation (3), that is, if the control is carried out with respect to the individual parameters.

If we set k_1, \dots, k_n equal to zero and reduce the determinant (4) to polynomial form, we obtain the coefficients of the characteristic equation, which are independent of K . If, in the expression (4), we set all K equal to zero with the exception of one, we can obtain the coefficients of the characteristic equation, which depend only on the given parameter, etc.³

FOOTNOTES

(p. 3) ¹ Furthermore, to reduce the determinant to triangular form, one can change the order of the characteristic equation at the cost of computational errors.

(p. 5) ² The coefficients $p_k(s)$ can be calculated from the following recursion formula:

$$p_k(s+1) = \frac{p_{k-1}(s) - s p_k(s)}{s+1}; \quad p_1(1) = 1; \quad s = 0, 1, 2, \dots, n; \quad k = 1, 2, \dots, n.$$

When we use an electronic digital computing machine to calculate the coefficients, a more convenient formula is

$$p_k(s+1) = (-1)^{s+k+1} \frac{\sum_{v=k-1}^s |p_{k-1}(v)|}{s+1}; \quad s = 0, 1, 2, \dots, n; \quad k = 1, 2, \dots, n.$$

Here,

$$p_0(0) = 1, \quad p_0(l) = 0, \quad l = 1, 2, \dots, n.$$

(p. 11) ³ If the coefficients of the characteristic equation depend on constants of the time of the regulators, on the system, or on other parameters, the determination of the coefficients of the characteristic equation as a function of these parameters must necessarily be carried out with consideration of the fact that the degree of the characteristic polynomial will change.